"Lectures on the hyperreals",
by Robert Goldblatt (2d edn, 2012),
p. 231

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Completion by Enlargement

There are many mathematical structures that are "incomplete" because they lack certain elements, such as the limit of a Cauchy sequence, the sum of infinite series, the least upper bound of a set of elements, a point "at infinity", and so on. A variety of standard techniques exist for *completing* such structures by adding the missing elements.

Now, the enlargement of a structure in a nonstandard framework is a kind of completion, and we are going to explore ways in which enlargements give an alternative approach to standard completions. From this perspective there is some redundancy in the enlargement process because in a sense it "saturates" a structure with all the elements one could ever imagine adjoining to it. Some of these new elements are irrelevant to completion, while others may be distinct but indistinguishable in terms of their role in completing the original structure. Thus we need to factor out such redundancy, and as we shall see, standard completions can typically be obtained as quotients of certain kinds of enlargement.

18.1 Completing the Rationals

The set ${}^*\mathbb{Q}$ of hyperrationals contains infinitely close approximations of all real numbers. For if $r \in \mathbb{R}$, then by transfer

$$(\forall x \in {}^*\mathbb{R})\,[r < x \to (\exists q \in {}^*\mathbb{Q})\,(r < q < x)].$$

So, putting $x = r + \varepsilon$ with ε a positive infinitesimal implies that there is some $q \in {}^*\mathbb{Q}$ with $r < q < r + \varepsilon$ and hence $q \simeq r$. Thus $q \in {}^*\mathbb{Q} \cap \mathrm{hal}(r)$ and